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Note on Generalized Root Systems and Generalized Quantum Groups

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Abstract: We introduce applications of generalized root systems to generalized quantum groups.

1 Introduction (History)

In [HY08], the axiomatic definition of generalized root systems is introduced. It is improved in [CH09]. Those definitions use semigroup terminology, or categorical terminology. In [Y15] (see also [BY18]), those were rewritten without using categorical terminology. Weyl groupoids are naturally defined associated to the generalized root systems. Recall the Matsumoto theorem for the Coxeter group $\langle s_i | i \in I \rangle$, which tells that two reduced expression of the same element of the Coxeter group can be transformed from one to the other by repetition of changing expression by

$$(M2) \underbrace{s_i s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_j s_i \cdots}_{m_{ij}} \text{ with } i, j \in I, i \neq j \text{ and } m_{ij} := |\{(s_i s_j)^k | k \in \mathbb{Z}\}| < \infty.$$

Note that the defining relations of the Coxeter group are composed of (M2) and (M1) $s_i^2 = i$ ($i \in I$), called the Coxeter relations. In [HY08], it is shown that Matsumoto-type theorem for the Weyl groupoids holds and that the defining relations of the Weyl groupoids are formed by the Coxeter-type relations. In AY18, we introduced Nil-Hecke algebras and a Bruhat order of the Weyl groupoids (see Theorem 3 of this paper). As applications of the Weyl groupoids, we have achieved:

(1) Although Weyl groupoids had not been introduced, the Serre-type defining relations of the affine Lie superalgebras are obtained in [Y99]. This reproved the Serre-type defining relations of the finite dimensional simple Lie superalgebras of type $A-G$ obtained in [Y94]. In [Y94] and [Y99], we also got the Serre-type defining relations of the finite and affine type quantum superalgebras in some way. In [AAY11], we got the Serre-type defining relations of finite-super- $ABCD$ -type Nichols algebras of diagonal-type (including multi-parameter finite- $ABCD$ -type quantum superalgebras).

(2) We got the Drinfeld second realizations of the $A^{(1)}(m, n)$ -type (resp, $D^{(1)}(2, 1; x)$ -type) quantum superalgebras in [Y99] (resp. [HSTY08]).

(3) We got the Shapovalov determinant formula

$$\det \text{Shap}_\lambda^{x, \pi} = \prod_{\alpha \in R^+(\chi, \pi)} \prod_{t_\alpha=1}^{\infty} (-\hat{\rho}^{x, \pi}(\alpha) \chi(\alpha, \alpha)^{-t_\alpha} K_\alpha + L_\alpha)^{|\mathcal{P}_\lambda^{x, \pi}(\alpha; t_\alpha)|}.$$

of the finite-type generalized quantum groups $U(\chi, \pi)$ in [HY10, Theorem 7.3], where $\text{Shap}_\lambda^{x, \pi}$ means the Shapovalov matrix of the weight λ and where see [BY18, Theorem 7.5] for detail of the right hand side; we needed the condition that $\chi(\alpha, \alpha) \neq 0$ for $\alpha \in R^+(\chi, \pi)$. Finite-type $U(\chi, \pi)$ can be finite-type quantum groups and finite-type quantum superalgebras and their Lusztig's small quantum groups.

(4) We got the classification theorem of the finite-dimensional irreducible representations of the finite-type non-finite dimensional $U(\chi, \pi)$ in [AYY15] over zero-characteristic field, and we also recover in [AYY15] the Kac's list of the classification theorem of the finite-dimensional simple modules of the finite-dimensional simple Lie superalgebras of type $A-G$.

(5) We got the explicit formula of the universal R -matrix of the finite-type $U(\chi, \pi)$ in [AY15].

(6) We got the Harish-Chandra theorem for $U(\chi, \pi)$ in (3) in [BY18], see Section 3 of this paper.

(7) We got the Kostant-Lusztig \mathbb{A} -form of the finite-type multi parameter quantum groups in [JMY17].

2 Generalized Root Systems

Let I be a non-empty finite set. Let V be a \mathbb{R} -linear space with a \mathbb{R} -basis $\{v_i | i \in I\}$, so $\dim_{\mathbb{R}} V = |I|$ and $V =: \oplus_{i \in I} \mathbb{R} v_i$. Let $V_{\mathbb{Z}} =: \oplus_{i \in I} \mathbb{Z} v_i$. Then $V_{\mathbb{Z}}$ be a free \mathbb{Z} -module with a \mathbb{Z} -basis $\{v_i | i \in I\}$, and $\text{rank}_{\mathbb{Z}} V_{\mathbb{Z}} = |I|$. Let $\mathcal{P}(V_{\mathbb{Z}})$ be a power set of $V_{\mathbb{Z}}$. Let $\mathcal{B}(V_{\mathbb{Z}})$ be a set of all \mathbb{Z} -bases of $V_{\mathbb{Z}}$, so $\mathcal{B}(V_{\mathbb{Z}}) \subset \mathcal{P}(V_{\mathbb{Z}})$.

Let $R \in \mathcal{P}(V_{\mathbb{Z}})$. Assume $R \neq \emptyset$. For $B \in \mathcal{B}(V_{\mathbb{Z}})$, let $R^{B, +} := R \cap \text{Span}_{\mathbb{Z}_{\geq 0}} B$ and $R^{B, -} := R \cap \text{Span}_{\mathbb{Z}_{\leq 0}} B$. We say that B is a *base* of R if the following (x) and

(y) hold:

- (x) $R = R^{B,+} \cup R^{B,-}$.
- (y) $\forall \alpha \in B, \mathbb{Z}\alpha \cap R = \{\alpha, -\alpha\}$.

Let $\mathcal{B}(R)$ be a set of all bases of R , so $\mathcal{B}(R) \subset \mathcal{B}(V_{\mathbb{Z}})$.

Let $R \in \mathcal{P}(V_{\mathbb{Z}})$. Assume $R \neq \emptyset$. Let \mathbb{B} be a non-empty subset of $\mathcal{B}(R)$. We say that (R, \mathbb{B}) is a *generalized root system* [HY08] (see also [BY18, Y15]) if

$$\forall B \in \mathbb{B}, \forall \alpha \in B, \exists \tau_{\alpha}(B) \in \mathbb{B}, R^{\tau_{\alpha}(B),+} \cap R^{B,-} = \{-\alpha\}.$$

For $R \in \mathcal{P}(V_{\mathbb{Z}})$, let $\mathcal{M}(R)$ be the set of all maps from I to R .

Theorem 2.1. [BY18, HY08, Y15]. *Let (R, \mathbb{B}) be a generalized root system. Then there exist a non-empty subset $\check{\mathbb{B}}$ of $\mathcal{M}(R)$ and bijections $\tau_i : \check{\mathbb{B}} \rightarrow \check{\mathbb{B}}$ ($i \in I$) satisfying (a)-(c) below.*

- (a) *The map $\varphi : \check{\mathbb{B}} \rightarrow \mathcal{P}(R)$ defined by $\varphi(\pi) := \pi(I)$ is injection, where $\mathcal{P}(R)$ is the power set of R .*
- (b) $\varphi(\check{\mathbb{B}}) = \mathbb{B}$.
- (c) $\forall \pi \in \check{\mathbb{B}}, \forall i \in I, (\tau_i(\pi))(I) = \tau_{\pi(i)}(\pi(I))$.

In particular, for $\pi \in \check{\mathbb{B}}$ and $i, j \in I$, there exist $N_{ij}^{\pi} \in \mathbb{Z}$ such that $(\tau_i(\pi))(j) = \pi(j) + N_{ij}^{\pi}\pi(i)$, which implies that $N_{ii}^{\pi} = -2$ and $N_{ij}^{\pi} \in \mathbb{Z}_{\geq 0}$ ($j \neq i$). Moreover, $(\tau_i)^2 = \text{id}_{\check{\mathbb{B}}}$ and $N_{ij}^{\tau_i(\pi)} = N_{ij}^{\pi}$.

Let $\pi \in \check{\mathbb{B}}$. For a map $f : \mathbb{N} \rightarrow I$ and $t \in \mathbb{Z}_{\geq 0}$, define $\pi_{f,t} \in \check{\mathbb{B}}$ by $\pi_{f,0} := \pi$ and $\pi_{f,t} := \tau_{f(t)}(\pi_{f,t-1})$ ($t \in \mathbb{N}$).

Lemma 2.2. *Assume $|R| < \infty$. Let $k := \lfloor \frac{|R|}{2} \rfloor$. Then $R^{\pi(I),+} = \{\pi_{f,t-1}(f(t)) | t \in \mathbb{N}, 1 \leq t \leq k\}$ and $R^{\pi(I),-} = -R^{\pi(I),+}$.*

Let \mathbb{J} be the set of all maps $f : \mathbb{N} \rightarrow I$.

Let (R, \mathbb{B}) be a generalized root system. For $\pi \in \check{\mathbb{B}}$ and $i \in I$, define the \mathbb{Z} -module automorphism $s_i^{\pi} : V \rightarrow V$ by $s_i^{\pi}(v_j) := v_j + N_{ij}^{\pi}v_i$. For $\pi \in \check{\mathbb{B}}$ and $f \in \mathbb{J}$, let $\pi_{f,0} := \pi$ and $\pi_{f,t} := \tau_{f(t)}(\pi_{f,t-1})$, and let $1^{\pi}s_{f,0} := \text{id}_V$ and $1^{\pi}s_{f,t} = (1^{\pi}s_{f,t-1}) \circ s_{f(t)}^{\pi_{f,t-1}}$. Let

$$\ell_{f,t}^{\pi} := \min\{r \in \mathbb{Z}_{\geq 0} | \exists g \in \mathbb{J}, 1^{\pi}s_{g,r} = 1^{\pi}s_{f,t}\}.$$

Theorem 2 [HY08]. *We have*

$$\ell_{f,t}^{\pi} = |R^{\pi_{f,t}(I),+} \cap R^{\pi,-}|.$$

In particular, if $1^\pi s_{f,t} = 1^\pi s_{g,r}$, then $\pi_{f,t} = \pi_{g,r}$.

Let (R, \mathbb{B}) be a generalized root system. Let $\check{\mathbb{B}}_\lambda$ ($\lambda \in \Lambda$) be non-empty subsets of $\check{\mathbb{B}}$ such that

- (i) $\cup_{\lambda \in \Lambda} \check{\mathbb{B}}_\lambda = \mathbb{B}$ and $\check{\mathbb{B}}_\lambda \cap \check{\mathbb{B}}_\mu = \emptyset$ ($\lambda \neq \mu$).
- (ii) For $\lambda \in \Lambda$ and $\pi, \pi' \in \check{\mathbb{B}}_\lambda$, defining the \mathbb{Z} -module automorphism $p : V \rightarrow V$ by $p(\pi(i)) := \pi'(i)$ ($i \in I$), we have $p(R^{\pi(I),+}) = R^{\pi'(I),+}$.
- (iii) $\forall \lambda \in \Lambda, \forall i \in I, \exists \mu \in \Lambda, \tau_i(\check{\mathbb{B}}_\lambda) = \check{\mathbb{B}}_\mu$.

For $\pi, \pi' \in \check{\mathbb{B}}$, we write $\pi \equiv \pi'$ if $\{\pi, \pi'\} \subset \check{\mathbb{B}}_\lambda$ for some $\lambda \in \Lambda$.

Theorem 3 [AY18]. *Assume that $\ell_{f,k}^\pi = k$ and $1^\pi s_{f,k} = 1^\pi s_{g,k}$. Assume that there exists a non-empty proper subset $S = \{i_1, \dots, i_x\}$ ($i_{t-1} < i_t$) of $\{1, \dots, k\}$ such that $\pi_{f,t-1} \equiv \pi_{f,t}$ ($t \in \{1, \dots, k\} \setminus S$) and $\ell_{f',x}^\pi = x$, where $f' \in \mathbb{J}$ is such that $f'(1) := i_1, \dots, f'(x) := i_x$. Then there exists a non-empty proper subset $T = \{j_1, \dots, j_x\}$ ($j_{t-1} < j_t$) of $\{1, \dots, k\}$ such that $\pi_{g,t-1} \equiv \pi_{g,t}$ ($t \in \{1, \dots, k\} \setminus T$) and $1^\pi s_{f',x} = 1^\pi s_{g',x}$, where $g' \in \mathbb{J}$ is such that $g'(1) := j_1, \dots, g'(x) := j_x$. (Note that $|S| = x = |T|$.)*

3 Generalized Quantum Groups

Let \mathbb{K} be an algebraically closed field. Let $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. Let $\chi : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{K}^\times$ be a map such that $\chi(\lambda, \mu + \nu) = \chi(\lambda, \mu)\chi(\lambda, \nu)$ and $\chi(\lambda + \mu, \nu) = \chi(\lambda, \nu)\chi(\mu, \nu)$ for all $\lambda, \mu, \nu \in V_{\mathbb{Z}}$. Let $\pi \in \check{\mathbb{B}}$. Let $U = U(\chi, \pi)$ be the \mathbb{K} -algebra (with 1) satisfying the following conditions (U1)-(U5). Existence and uniqueness of U is well-known.

(U1) U is generated by the elements K_λ, L_λ ($\lambda \in V_{\mathbb{Z}}$) and E_i, F_i ($i \in I$) satisfying the equations $K_0 = L_0 = 1$, $K_\lambda K_\mu = K_{\lambda+\mu}$, $L_\lambda L_\mu = L_{\lambda+\mu}$, $K_\lambda L_\mu = L_\mu K_\lambda$, $K_\lambda E_i K_{-\lambda} = \chi(\lambda, \pi(i))E_i$, $K_\lambda F_i K_{-\lambda} = \chi(\lambda, -\pi(i))F_i$, $L_\lambda E_i L_{-\lambda} = \chi(-\pi(i), \lambda)E_i$, $L_\lambda F_i L_{-\lambda} = \chi(\pi(i), \lambda)F_i$, $E_i F_j - F_j E_i = \delta_{ij}(-K_{\pi(i)} + L_{\pi(i)})$.

(U2) There exist the \mathbb{K} -subspaces $U_\lambda = U(\chi, \pi)_\lambda$ ($\lambda \in V_{\mathbb{Z}}$) of U such that $U = \oplus_{\lambda \in V_{\mathbb{Z}}} U_\lambda$, $U_\lambda U_\mu \subset U_{\lambda+\mu}$, $K_\lambda \in U_0$, $L_\lambda \in U_0$, $E_i \in U_{\pi(i)}$, $F_i \in U_{-\pi(i)}$.

(U3) Let $U^0 = U^0(\chi, \pi)$ (resp. $U^+ = U^+(\chi, \pi)$), resp. $U^- = U^-(\chi, \pi)$ be the \mathbb{K} -subalgebra (with 1) generated by $K_\lambda L_\mu$ ($\lambda, \mu \in V_{\mathbb{Z}}$) (resp. E_i ($i \in I$), resp. F_i ($i \in I$)). Define the \mathbb{K} -linear map $j_1 : U^- \otimes U^0 \otimes U^+ \rightarrow U$ by $j_1(Y \otimes Z \otimes X) := YZX$. Then j_1 is a \mathbb{K} -linear isomorphism.

(U4) Define the \mathbb{K} -linear map $j_2 : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow U^0$ by $j_2(\lambda, \mu) := K_\lambda L_\mu$. Then j_2 is

injective, and $j_2(V_{\mathbb{Z}} \times V_{\mathbb{Z}})$ is a \mathbb{K} -basis of U^0 .

(U5) We have $\{X \in U^+ | E_i X = X E_i (\forall i \in I)\} = U^+ \cap U_0$ and $\dim_{\mathbb{K}}\{Y \in U^- | F_i Y = Y F_i (\forall i \in I)\} = U^- \cap U_0$.

Let $V_{\mathbb{Z}}^{\pi,+} := \oplus_{i \in I} \mathbb{Z}_{\geq 0} \pi(i) (\subset V_{\mathbb{Z}})$. For $\lambda \in V_{\mathbb{Z}}$, let $U_{\lambda}^+ = U^+(\chi, \pi)_{\lambda} := U^+ \cap U_{\lambda}$ and $U_{\lambda}^- = U^-(\chi, \pi)_{\lambda} := U^- \cap U_{\lambda}$. Then $U^+ = \oplus_{\lambda \in V_{\mathbb{Z}}^{\pi,+}} U_{\lambda}^+$, $U^- = \oplus_{\lambda \in V_{\mathbb{Z}}^{\pi,+}} U_{\lambda}^-$, and $U_0^+ = U_0^- = \mathbb{K}1$. We also have $U_{\pi(i)}^+ = \mathbb{K}E_i$ and $U_{-\pi(i)}^- = \mathbb{K}F_i$ ($i \in I$).

For $n \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{K}^{\times}$, let $(n)_t := \sum_{x=1}^n t^{x-1}$ and $(n)_t! := \prod_{m=1}^n (m)_t$. For a set X and a map $f : X \rightarrow \mathbb{N}$, let $\mathfrak{M}(X, f) := \{(x, k) \in X \times \mathbb{N} | \forall x \in X, 1 \leq k \leq f(x)\}$ and define the map $p^{(X, f)} : \mathfrak{M}(X, f) \rightarrow X$ by $p^{(X, f)}(x, k) := x$. For $Y \in \mathcal{P}(V_{\mathbb{Z}})$ and a map $f : Y \rightarrow \mathbb{N}$, let $\mathfrak{P}^{(Y, f)}$ be the set of maps $g : \mathfrak{M}(Y, f) \rightarrow \mathbb{Z}_{\geq 0}$ such that $(g(y))_{\chi(p^{(Y, f)}(y), p^{(Y, f)}(y))!} \neq 0$ for all $y \in \mathfrak{M}(Y, f)$ and such that $|g^{-1}(\mathbb{N})| < \infty$. For $\lambda \in V_{\mathbb{Z}}^{\pi,+}$, $Z \in \mathcal{P}(V_{\mathbb{Z}}^{\pi,+})$ and a map $f : Z \rightarrow \mathbb{N}$, let $\mathfrak{P}_{\lambda}^{(Z, f)} := \{g \in \mathfrak{P}^{(Z, f)} | \sum_{z \in \mathfrak{M}(Z, f)} g(z) p^{(Z, f)}(z) = \lambda\}$.

Theorem 3.1. (*Kharchenko's PBW theorem [Kha99]*) *There exists a unique pair $((R^+(\chi, \pi), \varphi_+^{\chi, \pi})$ of $R^+(\chi, \pi) \in \mathcal{P}(V_{\mathbb{Z}}^{\pi,+})$ and a map $\varphi_+^{\chi, \pi} : Z \rightarrow \mathbb{N}$ such that $\dim U^+(\chi, \pi)_{\lambda} = \dim U^-(\chi, \pi)_{-\lambda} = |\mathfrak{P}_{\lambda}^{(R^+(\chi, \pi), \varphi_+^{\chi, \pi})}|$ for all $\lambda \in V_{\mathbb{Z}}^{\pi,+}$.*

Let $R(\chi, \pi) := R^+(\chi, \pi) \cup (-R^+(\chi, \pi)) (\in \mathcal{P}(V_{\mathbb{Z}}))$.

Theorem 3.2. (*Heckenberger's Weyl groupoid theorem [Hec06]*) *If $|R^+(\chi, \pi)| < \infty$, then $R(\chi, \pi)$ is a generalized root system and $\varphi_+^{\chi, \pi}(\alpha) = 1$ for all $\alpha \in R^+(\chi, \pi)$.*

Define the \mathbb{K} -linear map $\mathfrak{Sh}^{\chi, \pi} : U(\chi, \pi) \rightarrow U^0(\chi, \pi)$ by $\mathfrak{Sh}^{\chi, \pi}|_{U(\chi, \pi)^0} = \text{id}_{U(\chi, \pi)^0}$ and $\mathfrak{Sh}^{\chi, \pi}(\text{Span}_{\mathbb{K}}(U^-(\chi, \pi)_{-\lambda} U^0(\chi, \pi) U^+(\chi, \pi)_{\mu}) = \{0\}$ for $\lambda, \mu \in V_{\mathbb{Z}}^{\pi,+}$ with $\lambda + \mu \neq 0$.

Let $\omega : V_{\mathbb{Z}} \rightarrow \mathbb{K}^{\times}$ be a map such that $\omega(\lambda + \mu) = \omega(\lambda)\omega(\mu)$ for all $\lambda, \mu \in V_{\mathbb{Z}}$. Let $\mathfrak{Z}_{\omega}(\chi, \pi) := \{Z \in U(\chi, \pi)_0 | \forall \lambda \in V_{\mathbb{Z}}, \forall X \in U(\chi, \pi)_{\lambda}, ZX = \omega(\lambda)XZ\}$. Let $\mathfrak{H}\mathfrak{C}_{\omega}^{\chi, \pi} := \mathfrak{Sh}^{\chi, \pi}|_{\mathfrak{Z}_{\omega}(\chi, \pi)}$. Define the map $\hat{\rho}^{\chi, \pi} : V_{\mathbb{Z}} \rightarrow \mathbb{K}^{\times}$ by $\hat{\rho}^{\chi, \pi}(\sum_{i \in I} k_i \pi(i)) := \prod_{i \in I} \chi(\pi(i), \pi(i))^{k_i}$, where $k_i \in \mathbb{Z}$.

Lemma 3.3. ([BY18, Lemma 9.2]) *$\mathfrak{H}\mathfrak{C}_{\omega}^{\chi, \pi}$ is injective.*

Assume $|R^+(\chi, \pi)| < \infty$ and assume $\chi(\alpha, \alpha) \neq 1$ for all $\alpha \in R^+(\chi, \pi)$. Let $\beta \in R^+(\chi, \pi)$. Let $q_{\beta} := \chi(\beta, \beta)$. Let $c_{\beta} := 0$ if $q_{\beta}^n \neq 1$ for all $n \in \mathbb{N}$, and let $c_{\beta} := \min\{m \in \mathbb{N} | q_{\beta}^m = 1\}$ if $q_{\beta}^m = 1$ for some $m \in \mathbb{N}$. Let $\mathfrak{B}_{\omega}^{\chi, \pi}(\beta)$ be the \mathbb{K} -subspace of $U^0(\chi, \pi)$ formed by the elements $\sum_{\lambda, \mu \in V_{\mathbb{Z}}} a_{(\lambda, \mu)} K_{\lambda} L_{\mu}$ with $a_{(\lambda, \mu)} \in \mathbb{K}$ satisfying the following conditions (e1)-(e4). For $\lambda, \mu \in V_{\mathbb{Z}}$, let $\omega_{\lambda, \mu; \beta}^{\chi, \pi} := \frac{\omega(\beta)\chi(\beta, \mu)}{\chi(\lambda, \beta)}$.

(e1) For $\lambda, \mu \in V_{\mathbb{Z}}$ and $t \in \mathbb{Z}$, if $c_{\beta} = 0$ and $q_{\beta}^t = \omega_{\lambda, \mu; \beta}^{\chi, \pi}$, then $a_{(\lambda+t\beta, \mu-t\beta)} = \hat{\rho}^{\chi, \pi}(\beta)^t a_{(\lambda, \mu)}$.

(e2) For $\lambda, \mu \in V_{\mathbb{Z}}$, if $c_{\beta} = 0$ and $q_{\beta}^t \neq \omega_{\lambda, \mu; \beta}^{\chi, \pi}$ for all $t \in \mathbb{Z}$, then $a_{(\lambda, \mu)} = 0$.

(e3) For $\lambda, \mu \in V_{\mathbb{Z}}$ and $t \in \mathbb{N}$ with $1 \leq t \leq c_{\beta} - 1$, if $c_{\beta} \geq 2$ and $q_{\beta}^t = \omega_{\lambda, \mu; \beta}^{\chi, \pi}$, then $\sum_{x \in \mathbb{Z}} a_{(\lambda + (c_{\beta}x + t)\beta, \mu - (c_{\beta}x + t)\beta)} \hat{\rho}^{\chi, \pi}(\beta)^{-(c_{\beta}x + t)} = \sum_{y \in \mathbb{Z}} a_{(\lambda + c_{\beta}y\beta, \mu - c_{\beta}y\beta)} \hat{\rho}^{\chi, \pi}(\beta)^{-c_{\beta}y}$.

(e4) For $\lambda, \mu \in V_{\mathbb{Z}}$, if $c_{\beta} \geq 2$ and $q_{\beta}^t \neq \omega_{\lambda, \mu; \beta}^{\chi, \pi}$ for all $t \in \mathbb{Z}$, then for all $t \in \mathbb{N}$ with $1 \leq t \leq c_{\beta} - 1$, $\sum_{x \in \mathbb{Z}} a_{(\lambda + (c_{\beta}x + t)\beta, \mu - (c_{\beta}x + t)\beta)} \hat{\rho}^{\chi, \pi}(\beta)^{-(c_{\beta}x + t)} = \sum_{y \in \mathbb{Z}} a_{(\lambda + c_{\beta}y\beta, \mu - c_{\beta}y\beta)} \hat{\rho}^{\chi, \pi}(\beta)^{-c_{\beta}y}$.

Let $\mathfrak{B}_{\omega}^{\chi, \pi} := \cap_{\beta \in R^+(\chi, \pi)} \mathfrak{B}_{\omega}^{\chi, \pi}(\beta)$.

Theorem 3.4. ([BY18, Theorem 10.4]) *Assume $|R^+(\chi, \pi)| < \infty$ and assume $\chi(\alpha, \alpha) \neq 1$ for all $\alpha \in R^+(\chi, \pi)$. Then $\text{Im} \mathfrak{H} \mathfrak{C}_{\omega}^{\chi, \pi} = \mathfrak{B}_{\omega}^{\chi, \pi}$.*

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